**Solutions:**

1. Define (by assumption) and.

Let , for some arbitrarily small (to be defined), so that .

Let. Both are analytic (as polynomials) inside and on the closed contour , and for :

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The inequality is because achieves the minimum value when is real. To see this, let , with . Then:

whence the minimum is attained at (), and is equal to , in view of the fact that .

We show that there exists such that and, additionally, it holds that:

Let (defined on real numbers). Observe that , and (by the assumption that ). This implies that there exists arbitrarily small , such that , whence satisfies for .

Since is the only root of , and , an application of Rouché’s theorem implies that has a single root in . Since can be chosen arbitrarily small, this shows further that has a unique root in

We show that the unique root is real using contradiction. Assume, by contradiction, that the unique root is not real. Since the characteristic polynomial has real coefficients, the complex conjugate of a root is also root. Furthermore, the complex conjugate has the same norm, which implies that has two roots in —contradiction.

1. Let . The characteristic equation is . follows from Little’s law, and in view of and the characteristic equation, we get:

Alternatively, we can also obtain this from the stationary distribution as:

where we have simply re-indexed , and used the definition of . From this, we can verify using the characteristic equation that:

1. We need to show that an random variable tends to a deterministic constant as . The *characteristic function* (i.e., generating function restricted to imaginary values) defined as , where is equal to:

Using the fact that for any (complex), this means that

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which is precisely the characteristic function of the deterministic value .

A classical result states equivalence of (pointwise) convergence of characteristic functions and convergence in distribution, which concludes the derivation.

An alternative derivation is to use Stirling’s formula to show that the pdf tends to a Dirac centered at , i.e.,

1. only if part: use formulas for two-class M/M/1//PR for a counter-example